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CONNECTED CERTIFIED DOMINATION IN THE CENTRAL GRAPH OF CERTAIN GRAPHS

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Abstract

In this study, we explore the connected certified domination number of central graphs derived from specific standard graphs. A certified dominating set is defined as a dominating set S of a graph $G = (V, E)$ where each vertex in S has either zero or at least two neighbors in $V(G) - S$. Additionally, a certified dominating set S of G is categorized as a connected certified dominating set if the subgraph induced by S, denoted as $G[S]$, is connected. The connected certified domination number, denoted as $\gamma_{cer}^{c}(G)$, is the minimum cardinality among all connected certified dominating sets.

Keywords: Dominating set, certified dominating set, certified domination number, central graphs.

AMS Subject Classification Number: 05C69

1. Introduction:

Let $G = (V, E)$ represent a finite, undirected graph that does not contain loops or multiple edges. The graph G has $n = |V|$ vertices and $m = |E|$ edges. A graph P_n is defined as a sequence of vertices $v_1, v_2, \ldots, v_n$, where the edges are $\{v_i v_{i+1}\}$, for $i = 1, 2, \ldots, n-1$. A cycle is a path from a vertex back to itself (So the first and last vertices are not distinct). A graph in which every pair of distinct vertices is adjacent is known as a complete graph K_n. A simple bipartite graph with bipartition (X, Y), where each vertex of X is connected to every vertex of Y, is referred to as a complete bipartite graph denoted by $K_{m,n}$. A star is a complete bipartite graph $K_{1,n}$. The join of $G + H$ of graphs G and H is the graph with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set $E(G + H) = E(G) \cup E(H) \cup uv; u \in V(G)$ and $v \in V(H)$. The order n fan graph is represented by $K_1 + P_n$ and is denoted as F_n and $F_{1,n}$. On the other hand, the order $n \geq 3$ wheel graph is represented by $K_1 + C_n$ and is denoted as W_n and $W_{1,n}$. A vertex with a degree of 1 is referred to as an end vertex, while a vertex that is connected to an end vertex is known as a support vertex. Please refer to [1] for further information.

Graph domination is a captivating field within graph theory that holds significant relevance in both Engineering and Science due to its wide-ranging applications. There are more than 300 domination parameters available in the literature. We recommend readers outstanding books [2], [3] on domination-related parameters.

Suppose that we are given a group of X officials and a group of Y civilians. There must be an official $x \in X$ for each civil $y \in Y$ who can attend x and every time any such y is attending x there must be also another civil $z \in Y$ that observes y. That is z must act as a kind of witness, to side step any mismanagement from y. What is the minimum number of connected officials required to ensure the provision of this service in the context of a specific social network? This aforementioned issue motivates us to propose the concept of connected certified domination.

The concept of certified domination was initially proposed by Dettlaff, Lemanska, Topp, Ziemann, and Zylinski [9], and subsequently explored in greater detail in [6, 7, 8]. It has many applications in real life situations. A. Ilyass and V.S. Goswami [10] introduced the notion of connected certified domination. This motivated we to study the connected certified number in central graphs of certain standard graphs such as complete, complete bipartite graph, path graph, cycle graph, wheel graph, fan graph and double star graph.

In their research, authors in [6, 7, 8] investigated the concept of certified domination number in graphs. This number is defined as follows:

Definition 1.1. Consider a graph $G = (V, E)$ with an order of n. We define a subset $S \subseteq V(G)$ as a certified dominating set of G if S is a dominating set of G and each vertex in S has either zero or at least two neighbors in $V - S$. The certified domination number, denoted by $\gamma_{cer}(G)$, represents the smallest size f certified dominating sets in G.

Definition 1.2. Consider a connected graph $G = (V, E)$ with an order of n. A certified dominating set $S \subseteq V(G)$ is referred to as a connected certified dominating set of G if the induced subgraph $G[S]$ is connected. The connected certified domination number, denoted as $\gamma_{cer}^{c}(G)$, represents the minimum cardinality of a connected dominating set of G.

2. Preliminaries

Theorem 2.1. [9] Every certified dominating set of a graph G with order $n \geq 2$ includes its support vertices.

Theorem 2.2. [9] For every graph G with an order of n, the value of $\gamma_{cer}(G)$ lies between 1 and n.

Observation 2.3. [10]

1) If $K_{m,n}$ be a complete bipartite graph, then $\gamma_{cer}^{c}(K_{m,n}) = 2$ for $3 \leq m \leq n$.
2) If $K_{1,n-1}$ be a star graph, then $\gamma_{cer}^{c}(K_{1,n}) = 1$ for $n \geq 2$.
3) If W_n be a wheel graph, then $\gamma_{cer}^{c}(W_n) = 1$.
4) If $S_{1,n,n}$ be a double star graph, then $\gamma_{cer}^{c}(S_{1,n,n}) = 2$, where $n \geq 2$.

Observation 2.4. [10]

1) If K_n is a complete graph, then $\gamma_{cer}^{c}(K_n) = 1$ for $n \geq 3$.
2) If P_n is a path graph, then $\gamma_{cer}^{c}(P_n) = n$ for $n \geq 1$.
3) If C_n is a cycle graph, then $\gamma_{cer}^{c}(C_n) = n$ for $n \geq 3$.
4) If F_n is fan graph, then $\gamma_{cer}^{c}(F_n) = 1$ for $n \geq 2$.

Observation 2.5. [10]

For every connected graph G, the value of $\gamma_{cer}(G)$ is always less than or equal to $\gamma_{cer}^{c}(G)$.

3. Central graphs

Definition 3.1 [4, 5]

The central graph $C(G)$ of a graph G of order n and size m is the graph of order $n+m$ and $\binom{n}{2}+m$ which is obtained by subdividing each edge of G exactly once and joining all the non-adjacent vertices of G in $C(G)$.

Theorem 3.2.

Let G be a connected graph of order $n \geq 2$. Then, $2 \leq \gamma_{cer}^{c}(C(G)) \leq n+m$.

Proof.

Let G be a connected graph of order $n \geq 3$. Let the vertex set of G be $V(G) = \{v_1, v_2, \ldots, v_n\}$. Then $V(C(G)) = V \cup W$, where $W = \{e_{ij}; v_i v_j \in E(G)\}$. First we prove that the lower bound $\gamma_{cer}^{c} \geq 2$. By theorem 2.1 and observation 2.5, it is enough to prove $\gamma_{cer}^{c}(C(G)) \neq 1$. Suppose $\gamma_{cer}^{c}(C(G)) = 1$. Then there exists a vertex $v \in V(C(G))$ such that $\deg_{C(G)}(v) = V(C(G)) - 1$. This shows that v is adjacent to every vertex in $C(G)$. This conclude that $C(G)$ must have a cycle of length 3, which is a contradiction because $C(G)$ be a triangle-free graph. Therefore $\gamma_{cer}^{c}(C(G)) \geq 2$. Next to prove the upper bound $\gamma_{cer}^{c}(C(G)) \leq n+m$. Since the set of all vertices of $C(G)$ forms a connected certified dominating set of G, it is clear that $\gamma_{cer}^{c}(C(G)) \leq n+m$.

Theorem 3.3.

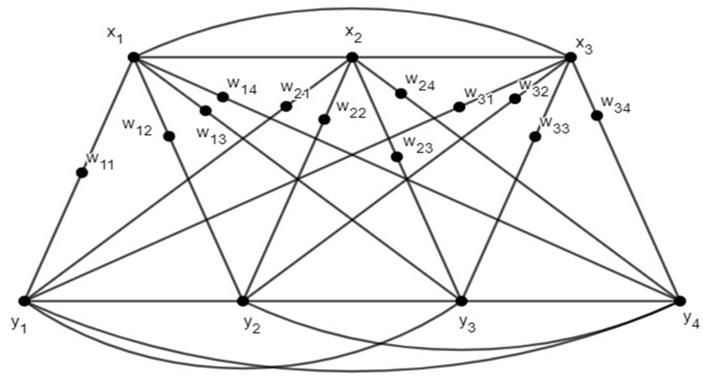
If $n \geq m \geq 3$, then $\gamma_{cer}^{c}\left(C(K_{m,n})\right) = m+2$.

Proof.

Let $G = K_{m,n}$ be a graph with $2 \leq m \leq n$. Let $X \cup Y$ be the partition of the vertex set of G, to the independent sets $X = \{x_i; 1 \leq i \leq m\}$ and $Y = \{y_j; 1 \leq j \leq n\}$. Then $X \cup Y \cup W$ is a partition of the vertex set of $C(G)$ such that $W = \{w_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\}$ and edge set $E(C(G)) = \{x_i x_j; 1 \leq i \leq j \leq m\} \cup \{y_i y_j; 1 \leq i \leq j \leq n\} \cup \{x_i w_{ij}, y_j w_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\}$. Let S be any connected certified dominating set of G. If $x_i \notin S$ and $y_j \notin S$ for some i and j, then it is clear that the subgraph induced by S in $C(G)$ is not connected, which is a contradiction. Thus we have either $X \subseteq S$ or $Y \subseteq S$. Without loss of generality we assume $X \subseteq S$. Since $C(G)[S]$ is connected and $N_{C(G)}[S] = V(C(G))$, that $w_{i1} \in S$ for some $1 \leq i \leq m$ or $y_k \in S$ for some $2 \leq k \leq n$. Thus, $\gamma_{cer}^{c}(C(G)) \geq m+1$. Next we prove $\gamma_{cer}^{c}(C(G)) \geq m+2$. Suppose $\gamma_{cer}^{c}(C(G)) = m+1$. Then $S = X \cup \{w_{i1}\}$ for some i or $S = X \cup \{y_k\}$ for some k. If $S = X \cup \{w_{i1}\}, then\ N_{C(G)}[y_j] \cap S = \phi$ for every $2 \leq j \leq n$, implies that y_j is not dominated by an vertices in S, which is a contradiction. If $S = X \cup \{y_k\}$, then the subgraph induced by S in $C(G)$ is not connected, again a contradiction. Thus, $\gamma_{cer}^{c}(C(G)) \geq m+2$. Now since, $S = X \cup \{y_1, w_{i1}\}$ is a connected certified dominating set of g with cardinality $m+2$. Hence $\gamma_{cer}^{c}(C(G)) = m+2$.

Example 3.4.

Consider $K_{3,4}$ with vertex set $V(K_{3,4}) = \{x_1, x_2, x_3, y_1, y_2, y_3, y_4\}$. Let $W = \{w_{ij}; 1 \leq i \leq 3, 1 \leq j \leq 4\}$. Then $V\left(C(K_{3,4})\right) = V(K_{3,4}) \cup W$ and $E\left(C(K_{3,2})\right) = \{x_i w_{ij}, y_j w_{ij}; 1 \leq i \leq 3, 1 \leq j \leq 4\}$ and the graph $C(K_{3,4})$ is shown in Figure 3.1.



$C(K_{3,4})$

Figure 3.1

It is clear that $S = \{x_1, x_2, x_3, w_{i1}, y_1\}$ is a minimum connected certified dominating set of $C(K_{3,4}) = 5$.

Theorem 3.5.

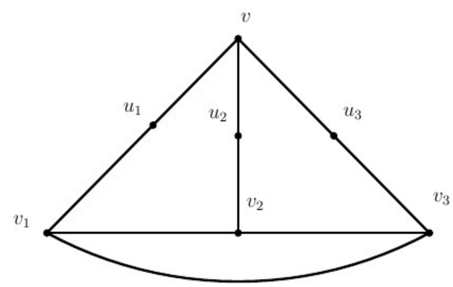
For any star graph $K_{1,n}$ of $n + 1 \geq 2$ vertices, $\gamma_{cer}^{c}(K_{1,n}) = 3$.

Proof.

Let $G = K_{1,n}$ be the star graph with vertex set $V(G) = \{v, v_1, v_2, \dots, v_n\}$, where v as a centre vertex and $\{v_1, v_2, \dots, v_n\}$ as its leaves. Let $\{u_1, u_2, \dots, u_n\}$ be the set of vertices that are subdivided the edges $\{vv_1, vv_2, \dots, vv_n\}$, respectively. Then, $V(C(G)) = \{v, v_i, u_i; 1 \leq i \leq n\}$ and $E(C(G)) = \{vv_i, u_i v_i; 1 \leq i \leq n\} \cup \{v_i v_j; 1 \leq i < j \leq n\}$. Here v dominates all the vertices $u_1, u_2, \dots, u_n$ and each v_i dominates all the other v_j in $C(G)$ for $1 \leq i < j \leq n$. Also, it is clear that $|N(v) \cap (V(G) - \{v\})| = |N(v)| = n$ and $|N(v_i) \cap (V(G) - \{u_i\}| = |N(v_i) \cap (V(G) - \{u_i\}| = |N(v_i) \cup \{u_i\}| = n$ for all $v_i \neq v$ in $C(G)$. Further more $n \geq 2$ shows that for some $i, S = \{v, v_i\}$ forms a certified dominating set of $C(G)$. But the subgraph induced by S in $C(G)$ is not connected. So that S is not a connected certified dominating set of $C(G)$ and so $\gamma_{cer}^{c}(C(G)) > 2$. Now, since $S_1 = S \cup \{u_i\}$ is a connected dominating set of $C(G)$, we conclude that $\gamma_{cer}^{c}(C(G)) = 3$.

Example 3.6.

Consider the graph $G = K_{1,3}$ with vertex set $V(G) = \{v, v_1, v_2, v_3\}$. Divide the edges vv_1, vv_2, vv_3 in G and take the new vertices as u_1, u_2, u_3 respectively. Thus $C(G)$ is obtained as is shown in Figure 3.2.



G

Figure 3.2

Here it is clear that $S = [v, u_1, v_1\}$ is a minimum connected certified dominating set of $C(G)$ and so $\gamma_{cer}^{c}\big(C(G)\big) = 3$.

Theorem 3.7.

For any path graph P_n of $n \geq 2$ vertices, $\gamma_{cer}^{c}\big(C(P_n)\big) = \begin{cases} 1 \; if \; n = 2 \\ n + m \; if \; n = 3{,}4 \\ 3 \; if \; n = 5 \\ \left\lfloor \frac{n}{2} \right\rfloor \; if \; n \geq 6 \end{cases}$.

Proof:

Let P_n be a path graph of $n \geq 2$ vertices with vertex set $V(P_n) = \{v_1, v_2, \dots, v_n\}$ where $v_i v_j \in E(P_n)$ if and only if $2 \leq j \leq i + 1 \leq n$. Then by the definition of central graph, that $C(P_n)$ has vertex set as $V\big(C(P_n)\big) = \{v_i, u_j; 1 \leq i \leq n, 1 \leq j \leq n - 1\}$, where $\{u_1, u_2, \dots, u_{n-1}\}$ be the set of vertices that divide the edges $v_1v_2, v_2v_3, \dots, v_{n-1}v_n$ in $C(P_n)$, respectively.

Let S be a connected certified dominating set in $C(P_n)$. If $n = 2$, then it is clear that $C(P_2)$ is isomorphic to the star graph $K_{1,2}$ and so by observation 2.3(2), $\gamma_{cer}^{c}\big(C(P_n)\big) = 1$.

If $n = 3$, then it is clear that $C(P_3)$ is isomorphic to the cycle graph C_5 and so by observation 2.4 (3), $\gamma_{cer}^{c}\big(C(P_3)\big) = 5 = m + n$.

For $n = 4$, we can easily check that $S' = \{v_1, v_2, v_3, v_4\}$ is a minimum connected dominating set of $C(P_4)$. But $\left|N_{C(P_4)}[v_1] \cap V(C(P_4) - S')\right| = 1$, implies that S' is not a connected certified dominating set of $C(P_4)$. If we take $S_1 = S \cup \{v_1\}$, that $\{v_2\}$ has exactly one neighbor in $V\big(C(P_4)\big) - S$. Thus $\gamma_{cer}^{c}\big(C(P_4)\big) \geq 7 = m + n$. Hence $\gamma_{cer}^{c}\big(C(P_4)\big) = m + n$.

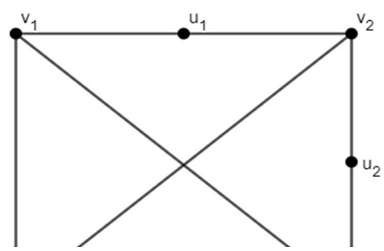
If $n = 5$, the one can easily verified that $S = \{v_1, v_3, v_5\}$ is a minimum connected certified dominating set of $C(P_5)$ and hence $\gamma_{cer}^{c}\big(C(P_5)\big) = 3$.

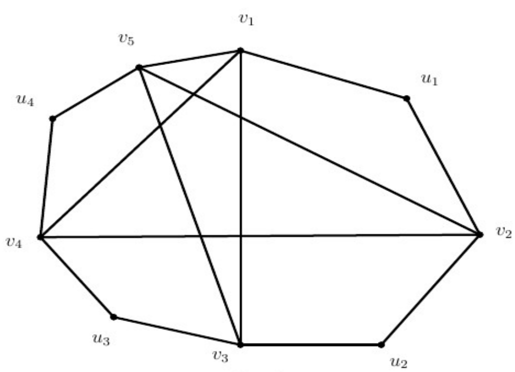
Now assume $n \geq 6$. By our construction, that $v_1u_1v_2u_2, \dots, u_{n-1}v_n$ induces a path of length $2n - 2$. Therefore, we need minimum $\frac{n}{2}$ vertices from $C(P_n)$, to dominates $V\left(C(P_n)\right)$. Thus, $\gamma_{cer}^{c}\big(C(P_n)\big) \geq \left\lfloor \frac{n}{2} \right\rfloor$ on the other hand, we can select $S = \{v_2, v_4, v_6, \dots, v_n\}$, which itself form a connected certified dominating set of $C(P_n)$ and hence we conclude that $\gamma_{cer}^{c}\big(C(P_n)\big) = \left\lfloor \frac{n}{2} \right\rfloor$.

Example 3.8

Here $S = V\big(C(P_4)\big)$ is a minimum connected certified dominating set of $C(P_4)$ and hence $\gamma_{cer}^{c}\big(C(P_4)\big) = |S| = 7$.

Consider $C(P_5)$ as the graph given in Figure 3.4.





$C(P_5)$

Figure 3.4

Here $S = \{v_1, v_3, v_5\}$ is a minimum connected certified dominating set of $C(P_5)$ and so $\gamma_{cer}^{c}\big(C(P_5)\big) = 3$.

Theorem 3.9

For any cycle graph C_n of $n \geq 3$ vertices , $\gamma_{cer}^{c}\big(C(C_n)\big) = \begin{cases} n + m \; if \; n = 3,4 \\ \left\lceil \frac{n}{2} \right\rceil \; if \; n \geq 5 \end{cases}$.

Proof.

Let C_n be a cycle graph of $n \geq 3$ vertices with vertex set $V(C_n) = \{v_1, v_2, \dots, v_n\}$, where $v_i v_j \in E(C_n)$ if and only if $j \leq i + 1(mod n)$. Then by the central graph definition, $C(C_n)$ has the vertex set $V\big(C(C_n)\big) = \{v_1, v_2, \dots, v_n, u_1, u_2, \dots, u_n\}$ where $u_1, u_2, \dots, u_n$ be the vertices that subdivide the edges $v_1 v_2, v_2 v_3, \dots, v_n v_1$ respectively in C_n.

Let S be a connected certified dominating set of $C(C_n)$. If $n = 3$, then it is clear that $C(C_3)$ is isomorphic to the cycle graph C_6 and hence by observation 2.4(3), $\gamma_{cer}^{c}\big(C(C_3)\big) = 6 = m + n$.

For $n = 4$, we can easily check that $S' = \{v_1, v_2, v_3\}$ is a certified dominating set of $C(C_4)$. But the subgraph induced by S' in $C(C_4)$ is not connected and so that S' is not a connected certified dominating set of $C(C_4)$. If we add any vertex from the set $\{u_1,\ u_2,\ u_3,\ u_4\}$ to S', then S' is not a connected certified dominating set of $C(C_4)$. Moreover we can easily observe that $\gamma_{cer}^{c}\big(C(C_4)\big) \geq 8 = m + n$. Since $V\big(C(C_4)\big)$ form a connected certified dominating set of $C(C_4)$, we conclude that $\gamma_{cer}^{c}\big(C(C_4)\big) = m + n$.

Now, assume that $n \geq 5$. By our construction that $v_1u_1v_2u_2, \dots, u_nv_1$ induces a cycle of length 2n, we need minimum $\frac{n}{2}$ vertices from $C(C_n)$ to dominates $C(C_n)$. Thus, $\gamma_{cer}^{c}\big(C(C_4)\big) \geq \left\lceil\frac{n}{2}\right\rceil$. To show that lower bound, we consider two cases.

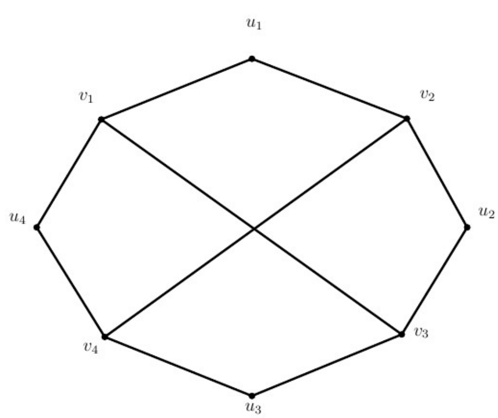
Case (i) Suppose n is odd. Take $n = 2k + 1$, where $k \geq 2$. Let $S = \{v_2, v_4, v_6, \dots, v_{2k}, v_{2k+1}\}$. Then one can easily verified that S is a connected certified dominating set of $C(C_{2k+1})$ and so $\gamma_{cer}^{c}\big(C(C_{2k+1})\big) = \left\lceil\frac{n}{2}\right\rceil$.

Case (ii) Suppose n is even. Take $n = 2k$, where $k \geq 3$. Here we consider $S = \{v_2, v_4, v_6, \dots, v_{2k}\}$. Then it is clear that S is a connected certified dominating set of $C(C_{2k})$ and so $\gamma_{cer}^{c}\big(C(C_{2k})\big) = \frac{n}{2}$.

Hence, $\gamma_{cer}^{c}\big(C(C_n)\big) = \left\lceil\frac{n}{2}\right\rceil$.

Example 3.10.

In Figure 3.5, C_4 is represented with vertex set $\{v_1, v_2, v_3, v_4\}$. The edges $v_1\, v_2, v_2v_3$, and v_4v_1 are divided, resulting in the creation of new vertices u_1, u_2, u_3, and u_4 in $C(C_4)$.



$C(C_4)$

Figure 3.5

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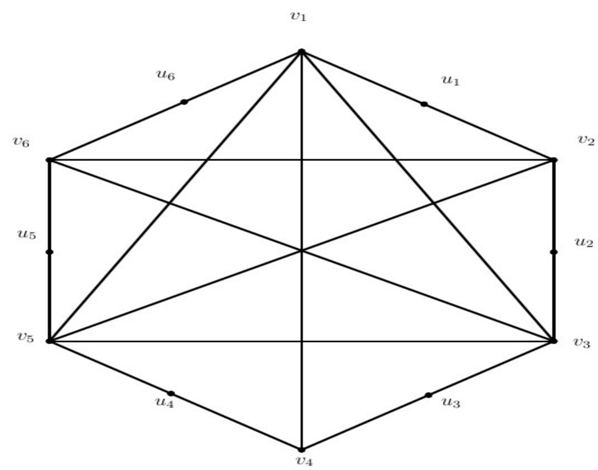
Here $S = \{v_1, v_2, v_3\}$ is a minimum certified dominating set of $C(C_4)$. But the subgraph induced by S in $C(C_4)$ is not connected.

Now it is clear that $S_1 = V\big(C(C_4)\big)$ is a minimum connected certified dominating seet of $C(C_4)$ and hence $\gamma_{cer}^{c}\big(C(C_4)\big) = 8$.

Consider $C(C_6)$ as the graph given in Figure 3.6.



$C(C_6)$
Figure 3.6

Here, set $S = \{v_2, v_4, v_6\}$ or $S = \{v_1, v_3, v_5\}$ is a minimum connected certified dominating set of $C(C_6)$ and hence $\gamma_{cer}^{c}\big(C(C_6)\big) = 3$.

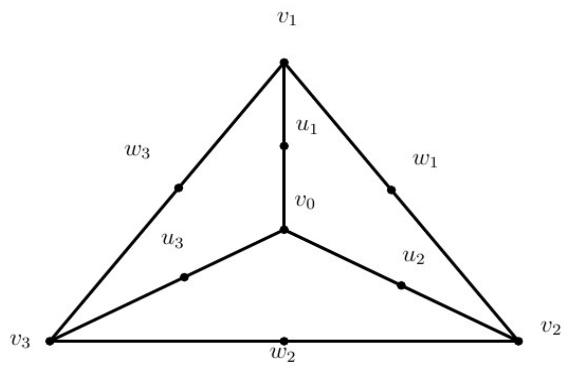
Theorem 3.11.

For any wheel graph W_n of $n + 1 \geq 4$ vertices, $\gamma_{cer}^{c}\big(C(W_n)\big) = \begin{cases} n+4 \; if \; n = 3 \\ n+2 \; if \; n = 4 \\ \left\lceil \frac{n+4}{2} \right\rceil \; if \; n \geq 5 \end{cases}$.

Proof.

Let W_n be a wheel graph of $n + 1 \geq 4$ vertices. Let the vertex set of W_n be $\{v_0, v_1, v_2, \dots, v_n\}$ and the edge set of W_n be $\{v_0 v_i, v_i v_{i+1}; 1 \leq i \leq n\}$. Then by the central graph definition, $C(W_n)$ has the vertex set $S\big(C(W_n)\big) = \{v_0, v_i, u_i, w_i; 1 \leq i \leq n\}$ and has the edge set $E\big(C(W_n)\big) = \{v_0 u_i, u_i v_i, v_i w_i; 1 \leq i \leq n\} \cup \{w_i v_{i+1}; 1 \leq i \leq n-1\} \cup \{w_n v_1\} \cup \big\{v_i v_j; 1 \leq j+1 \leq n-1\big\} \cup \{v_n v_1\}$.

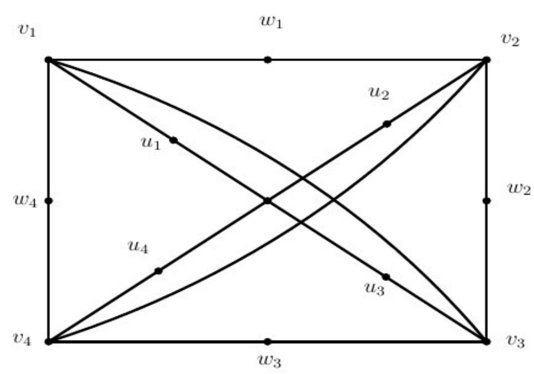
Let S be a connected certified dominating set of $C(W_n)$. If $n = 3$, then $C(W_3)$ is isomorphic to the subdivision graph of W_3, that is every edge of W_3 is divided exactly once and the graph is shown in Figure 3.7.



$C(W_3)$
Figure 3.7

If we take $S' = \{v_0, v_1, v_2, v_3\}$, then S' is a certified dominating set of $C(W_3)$. But the subgraph induced by S' in $C(W_3)$ is not connected. It is clear that $S = S' \cup \{u_1, u_2, u_3\}$ is a connected certified dominating set of $C(W_3)$. If we remove any vertex from S or there does not exist a connected certified dominating set of cardinality less than S. Therefore, that S is a minimum connected certified dominating set of $C(W_3)$ and so $\gamma_{cer}^{c}\big(C(W_3)\big) = 7$.

For $n = 4$, we can easily check that $S' = \{v_0, v_1, v_2, v_3\}$ is a certified dominating set of $C(W_4)$. But the subgraph induced by S'in $C(W_4)$ is not connected and so that S' is not a connected certified dominating set of $C(W_4)$. If we add any one vertex from $V\big(C(W_4)\big) - S'$ to S', then S' is not a connected certified dominating set of $C(W_4)$. Therefore, $\gamma_{cer}^{c}\big(C(W_4)\big) \geq 6$. Since $S = S' \cup \{w_2, u_2\}$ is a connected certified dominating set of $C(W_4)$ and so $\gamma_{cer}^{c}\big(C(W_4)\big) = 6$. The graph $C(W_4)$ is shown in Figure 3.8.



$C(W_4)$
Figure 3.8

Now assume that $n \geq 5$. since $W_n = C_n + K_1$, where $V(K_1) = \{v_0\}$ and $V(C_n) = \{v_1, v_2, \dots, v_n\}$. Thus that v_0 is adjacent to n-vertices in $C(W_n)$ and so $v_0 \in S$. Now we need to

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dominate the vertices in C_n and the vertices which subdivide the edges in C_n of $C(W_n)$. Consider two cases

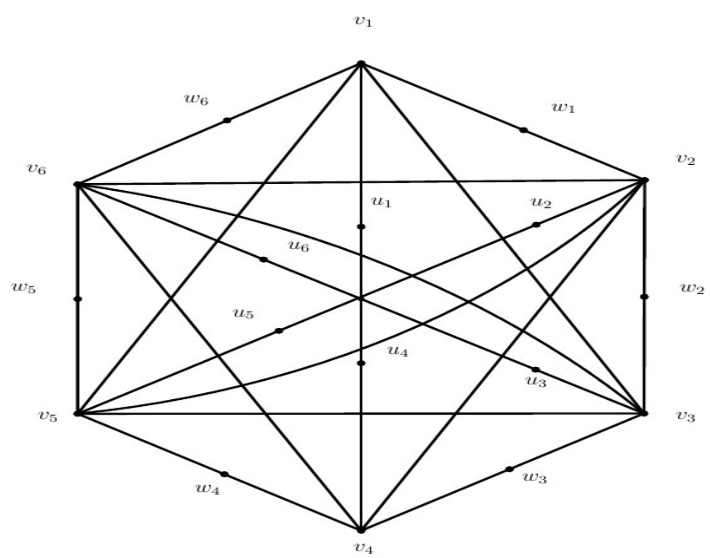
Case(i) Suppose n is odd. Let $n = 2k + 1$, where $k \geq 2$. Consider $S' = \{v_0, v_1, v_3, \dots, v_{2k-1}, v_{2k+1}\}$. Clearly S' is a certified dominating set in $C(W_n)$ with minimum cardinality. But subgraph induced by S' is not connected in $C(W_n)$. Let $S = S' \cup \{u_i\}$ for some i. Then S is a minimum connected certified dominating set of $C(W_n)$ and hence $\gamma_{cer}^{c}\big(C(W_n)\big) = |S| = \frac{n}{2} + 1 + 1 = \frac{n+4}{2}$.

Case(ii) Suppose n is even. Let $n = 2k$, where $k \geq 3$. Consider $S' = \{v_0, v_1, v_3, \dots, v_{2k-3}, v_{2k-1}\}$. Clearly S' is a minimum certified dominating set in $C(W_n)$. But the subgraph induced by S' in $C(W_n)$ is not connected. Take $S = S' \cup \{u_i\}$ for some i. Then it is clear that S is a minimum connected certified dominating set of $C(W_n)$ and hence $\gamma_{cer}^{c}\big(C(W_n)\big) = |S| = \left\lceil \frac{n}{2} \right\rceil + 1 + 1 = \left\lceil \frac{n+4}{2} \right\rceil$.

Then both the cases $\gamma_{cer}^{c}\big(C(W_n)\big) = \left\lceil \frac{n+4}{2} \right\rceil$.

Example 3.12

Consider the graph W_6. Let $V(W_i) = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6\}$. Then by above construction $C(W_n)$ has 19 vertices and the graph labeled in Figure 3.9.



$C(W_6)$

Figure 3.9

Here the set $S = \{v_0, v_1, v_3, v_5, u_1\}$ is a minimum connected certified dominating set of $C(W_8)$ and hence $\gamma_{cer}^{c}\big(C(W_6)\big) = 5$.

Theorem 3.13.

For any path graph F_n of $n + 1 \geq 4$ vertices, $\gamma_{cer}^{c}\big(C(F_n)\big) = \begin{cases} n + 3 \text{ if } n = 3 \\ \left\lceil \frac{n+5}{2} \right\rceil \text{ if } n \geq 4 \end{cases}$.

Proof .

Let F_n be a fan graph with $n+1 \geq 4$ vertices. Let the vertex set of F_n has $V(F_n) = \{v_0, v_1, v_2, \ldots, v_n\}$ and the edges set of F_n be $E(F_n) = \{v_0v_i, v_1v_2, v_2v_3, \ldots, v_{n-1}v_n; 1 \leq i \leq n\}$. Then by the central graph definition $C(F_n)$ has the vertex set $V\big(C(F_n)\big) = \{v_0, v_i, u_i; 1 \leq i \leq n\} \cup \{w_j: 1 \leq j \leq n-1\}$, where $u_1, u_2, \ldots, u_n$ be the vertices that subdivide the edges $v_0v_1, v_0v_2, v_0v_3, \ldots, v_0v_n$, respectively and w_j be the vertex that subdivide the edges v_jv_{j+1} for $1 \leq j \leq n-1$.

If $n = 3$, then $C(F_3)$ is the graph given in Figure 3.10. Let S be a minimum connected certified dominating set of $C(F_n)$.

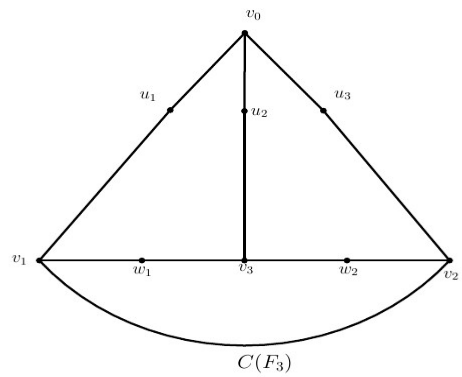


Figure 3.10

If we take $S' = \{v_0, v_1, v_2\}$, then one can easily verified that S' is a minimum certified dominating set of $C(F_3)$. However, the subgraph formed by S' in $C(F_3)$ is not connected due to the fact that $deg\ (v_0) = 3$ in $C(F_3)$. In order to establish connectivity, we must include at least one vertex from the set $\{u_1, u_2, u_3\}$ into S. Given that both v_0 and v_2 are already in S, we choose to add u_2 to S. Consequently, a path is formed between v_0, u_2 , and v_2, but it remains disconnected from v_1. To connected v_1, we need at least one vertex from the set $\{u_1, w_1, v_3\}$. Select u_1 to S. Then v_0 has exactly one neighbor $V\big(C(F_3)\big) - S$. Therefore u_3 must be in S. So u_3 has exactly one neighbor v_3 in $V\big(C(F_3)\big) - S$. Therefore v_3 must be in S. Then $S = S' \cup \{u_1, u_2, v_3\}$ be a connected certified dominating set of minimum cardinality. Hence $\gamma_{cer}^{c}\big(C(F_3)\big) = 6$.

Now, assume $n \geq 4$. Clearly $F_n = P_n + K_1$, where $V(K_1) = \{v_0\}$ and $V(P_n) = \{v_1, v_2, \ldots, v_n\}$. Therefore, v_0 is adjacent to every vertices in $V(P_n)$. Thus that v_0 is adjacent to n-vertices in $C(F_n)$ and so $v \in S$. Since the remaining $(2n-1)$ –vertices induces a path in $C(F_n)$, we have two cases to complete the result.

Case (i) Suppose n is odd. Let $n = 2k+1$, where $k \geq 2$. Consider $S' = \{v_0, v_1, v_3, \ldots, v_{2k-1}, v_{2k+1}\}$. Then it is clear that S' is a minimum certified dominating set in $C(F_n)$. But subgraph induced by S' is in $C(F_n)$ is not connected. Since v_0 is not adjacent to remaining vertices of S' in $C(F_n)$, we need to select at least one vertex from the set

$\{u_1, u_2, \dots, u_n\}$. Let select u_3. Then $S = S' \cup \{u_3\}$ be a minimum connected certified dominating set of $C(F_n)$ and hence $\gamma_{cer}^c\big(C(F_n)\big) = \left\lceil\frac{n+3}{2}\right\rceil + 1 = \left\lceil\frac{n+5}{2}\right\rceil$.

Case (ii) Suppose n is even. Let $n = 2k$, where $k \geq 2$. Consider $S' = \{v_0, v_1, v_3, \dots, v_{2k-1}, v_{2k}\}$. Clearly S' is a minimum certified dominating set in $C(F_n)$. But subgraph induced by S' is in $C(F_n)$ is not connected. As in case (i), we select u_3. Therefore $S = S' \cup \{u_3\}$ be a minimum connected certified dominating set of $C(F_n)$ and hence $\gamma_{cer}^c\big(C(F_n)\big) = \left\lceil\frac{n+3}{2}\right\rceil + 1 = \left\lceil\frac{n+5}{2}\right\rceil$.

Thus $\gamma_{cer}^c\big(C(F_n)\big) = \left\lceil\frac{n+5}{2}\right\rceil$ for $n \geq 4$.

Example 3.14.

Consider F_5 with vertex set $\{v_0, v_1, v_2, v_3, v_4, v_5\}$. Then the graph $C(F_5)$ with 15 vertices labeled in Figure 3.11.

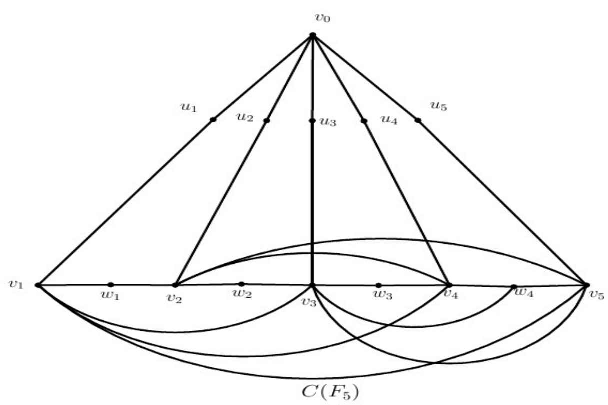


Figure 3.11

Here set $S = \{v_0, u_3, v_1, v_3, v_5\}$ is a minimum connected certified dominating set of $C(F_5)$ and so $\gamma_{cer}^c\big(C(F_n)\big) = 5 = \left\lceil\frac{5+5}{2}\right\rceil$.

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