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GAMMA GRAPH OF THE ZERO-DIVISOR GRAPH IN A FINITE COMMUTATIVE RING.

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Abstract

Consider the family of γ-sets of a zero-divisor graph $\Gamma(Z_n)$ of finite commutative ring Z_n and define the $\gamma-$graphs $\Gamma(Z_n)(\gamma) = (V(\gamma), E(\gamma))$ of $\Gamma(Z_n)$ to be the graph whose vertices $V(\gamma)$ correspond 1-to-1 with the γ - sets of G, and two γ sets, say S_1 and S_2, form an edge in $E(\gamma)$ if there exists a vertex $v \in S_1$ and a vertex $w \in S_2$ such that (i) v is adjacent to w and (ii) $S_1 = S_2 - \{w\} \cup \{u\}$ and $S_2 = S_1 - \{u\} \cup \{w\}$. Using this definition, we find some results. Also, Let R be a commutative ring. The gamma graph of the zero-divisor graph $\Gamma(R)$ of R is the graph $\gamma.\Gamma(R)$ with vertex set as the collection of all gamma sets of the zero-divisor graph $\Gamma(R)$ of R and two distinct vertices A and B are adjacent if and only if $|A \cap B| = \gamma(\Gamma(R)) - 1$ Using this definition, we have one basic property of $\gamma.\Gamma(R)$.

Keywords: Commutative Ring, Zero-divisors, Zero-divisor graph, gamma sets, gamma graph.
2020 subject classification: 05C25,05C69.

1. INTRODUCTION

The study of algebraic structures, using the properties of graph, become an exciting research topic in the past twenty years, leading to many fascinating results and questions. In the literature, there are many papers assigning graphs to rings, groups and semigroups. Let R be a commutative ring with identity and $Z(R)^*$ be the set of all non-zero zero-divisors of R. D.F. Anderson and P.S. Livingston[1], associates a graph called zero-divisor graph $\Gamma(R)$ to R with vertex set $Z(R)^*$ and for any two distinct $x, y \in Z(R)^*$, the vertices x and y are adjacent if and only if $xy = 0$ in R.

For $v \in V$, the associate class of v is defined as $A_v = \{uv: u \text{ is unit in } R\}$. Let $n = p_1^{k_1} p_2^{k_2} p_3^{k_3} \dots p_r^{k_r}$, where $p_1, p_2, \dots, p_r$ are primes with $p_1 < p_2 < \dots < p_r$ and $k_1, k_2, \dots, k_r$ are positive integers. Then the set of all non-zero zero-divisors in Z_n, the ring of congruent modulo n classes is given by $Z(Z_n)^* = \{\lambda_i p_i : 1 \le \lambda_i \le \frac{n}{p_i},\ 1 \le i \le r\}$.

A set $D \subseteq V$ of vertices of vertices in a graph $G = (V, E)$ is called a dominating set if for every vertex $u \in V - D$,there exists a vertex $v \in D$ such that v is adjacent to u. A dominating set D is minimal if no proper subset D is a dominating set. The domination number of a graph G, denoted by $\gamma(G)$, is the minimum cardinality of a minimal dominating set of G. A dominating set D in a graph G with cardinality γ is called $\gamma -$ set of G.

Subramanian and Sridharan introduced the concept γ- graph $\gamma.G$ and Fricke, et.al.[3] introduced the γ- graph $G(\gamma)$.

Definition 1.1: [2] Let D be the collection of $\gamma -$ sets in G. The gamma graph of G, denoted by $\gamma.G$, is the graph with vertex set D and any two vertices D_1 and D_2 are adjacent if $|D_1 \cap D_2| = \gamma(G) - 1$.

Definition 1.2: [3] Consider the family of γ-sets of a graph G and define the $\gamma -$graphs $G(\gamma) = (V(\gamma), E(\gamma))$ of G to be the graph whose vertices $V(\gamma)$ correspond 1-to-1 with the γ - sets of G, and two γ sets, say S_1 and S_2, form an edge in $E(\gamma)$ if there exists a vertex $v \in S_1$ and a vertex $w \in S_2$ such that

(i) v is adjacent to w and

(ii) $S_1 = S_2 - \{w\} \cup \{v\}$ and $S_2 = S_1 - \{v\} \cup \{w\}$.

Motivated by the above two definitions we have to introduced the gamma graph of a zero-divisor graph of a finite commutative rings.

The following results are used in the subsequent section.

Remark 1.3:[5] Let $n = {p_1}^{k_1}{p_2}^{k_2}{p_3}^{k_3} \dots {p_r}^{k_r}$, where $r \ge 1$, $p_1, p_2, \dots, p_r$ are primes with $p_1 < p_2 < \dots < p_r$ and $n \ne 2p, n \ne 3p, p > 3$ is prime. Then the number of $\gamma -$ sets in $\Gamma(Z_n)$ is $\prod_{i=1}^{r}(p_i - 1)$.

Remark 1.4. Let $n = {p_1}^{k_1}{p_2}^{k_2}{p_3}^{k_3} \dots {p_r}^{k_r}$, where $r \ge 1$, $p_1, p_2, \dots, p_r$ are primes with $p_1 < p_2 < \dots < p_r$ and $n = 3p, p > 3$ is prime. Then the number of $\gamma -$ sets in $\Gamma(Z_n)$ is $\prod_{i=1}^{r}(p_i - 1) + 1$.

Remark 1.5. Let $n - {p_1}^{k_1}{p_2}^{k_2}{p_3}^{k_3} \dots {p_r}^{k_r}$, where $r \ge 1$, $p_1, p_2, \dots, p_r$ are primes with $p_1 < p_2 < \dots < p_r$ and $n = 2p, p > 3$ is prime. Then the number of $\gamma -$ sets in $\Gamma(Z_n)$ is 1.

Corollary 1.6:[5] Let $n = p_1^{k_1} p_2^{k_2} p_3^{k_3} \dots p_r^{k_r}$, where $r \geq 1$, $p_1, p_2, \dots, p_r$ are primes with $p_1 < p_2 < \cdots < p_r$ and $n \neq 2p, p \geq 3$ is prime, then $\gamma(\Gamma(Z_n)) = r$.

Proposition 1.7:[4] Let (R, m) be a finite commutative ring local ring with $|R| = p^n$ for some prime p, $n > 1$ and let $x \in Z(R)^*$. Then $\{x\}$ is a γ- set of $\Gamma(R)$ if and only if $ann(x) = m$. Hence the number of distinct γ- sets in $\Gamma(R)$ is $p^k - 1$ for some $k < n$.

2. Gamma Graph $\Gamma(Z_n)(\gamma)$ of commutative ring Z_n.

Throughout this section, $n$is a fixed positive integer and not a prime number,
$Z_n = \{0,1,2,\dots,n-1\}$, $\Gamma(Z_n)$is the Zero-divisor graph of Z_n, $\Gamma(Z_n)(\gamma)$ is the gamma graph of $\Gamma(Z_n)$ and $V = V(\Gamma(Z_n)(\gamma))$ is the vertex set of $\Gamma(Z_n)(\gamma)$

Definition 2.1: Consider the family of γ-sets of a zero-divisor graph $\Gamma(Z_n)$ of finite commutative ring Z_n and define the $\gamma-$graphs $\Gamma(Z_n)(\gamma) = (V(\gamma), E(\gamma))$ of $\Gamma(Z_n)$ to be the graph whose vertices $V(\gamma)$ correspond 1-to-1 with the γ - sets of G, and two γ sets, say S_1 and S_2, form an edge in $E(\gamma)$ if there exists a vertex $v \in S_1$ and a vertex $w \in S_2$ such that

(i) v is adjacent to w and

(ii) $S_1 = S_2 - \{w\} \cup \{v\}$ and $S_2 = S_1 - \{v\} \cup \{w\}$.

With this definition, two γ-sets are said to be adjacent if they differ by one vertex and the two vertices defining this difference are adjacent $\Gamma(Z_n)$.

Example 2.2: Consider the ring Z_{25}.

$Z_{15} = \{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,20,21,22,23,24\}$

$Z(Z_{15})^* = \{5,10,15,20\}$

The zero-divisor graph $\Gamma(Z_{25})$:

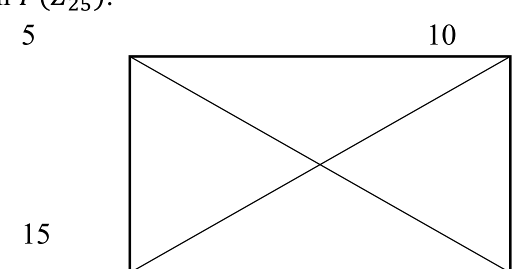

Fig.1 $\Gamma(Z_{25})$

The γ- sets are $S_1 = \{5\}$, $S_2 = \{10\}$, $S_3 = \{15\}$, $S_4 = \{20\}$.
Now, Let $v = 5 \in S_1$ and $w = 10 \in S_2$.

$S_2 - \{w\} \cup \{v\} = \{10\} - \{10\} \cup \{5\}$
$= \{5\}$
$= S_1$

$$S_1 - \{v\} \cup \{w\} = \{5\} - \{5\} \cup \{10\}$$
$$= \{10\}$$
$$= S_2$$

Also, {5} and {10} are adjacent.
Therefore, S_1 and S_2 are adjacent.
Similarly, S_1 is adjacent to S_3, S_4 and S_2 is adjacent to S_3, S_4 and S_3 is adjacent to S_4

Then we get the gamma graph $\Gamma(Z_{25})(\gamma)$ is of below form:

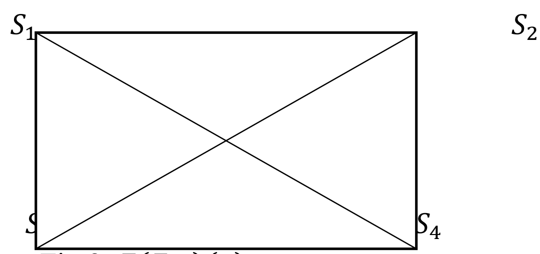

Fig 2. $\Gamma(Z_{25})(\gamma)$

Theorem 2.3: Let n be a positive integer and not a prime number. Then $\Gamma(Z_n)(\gamma) = K_1$ if and only if $n = 2^k$ or $n = 2p$, where $p \neq 2$ is a prime number, $k \geq 2$.

Proof: Suppose $n = 2^k$, $k \geq 2$.

Then $\Gamma(Z_n) = K_1$ and so $\gamma(\Gamma(Z_n)) = 1$.
Moreover there is only one γ-set.
Therefore, $\Gamma(Z_n)(\gamma) = K_1$.
Suppose that $n = 2p$, where $p \neq 2$ is a prime number.

Then $\Gamma(Z_n) = K_{1,p-1}$ and so $\gamma(\Gamma(Z_n)) = 1$.
There is only one γ-set.
Therefore, $\Gamma(Z_n)(\gamma) = K_1$.
Converse part is obviously from Remark 1.3.

Theorem 2.4. Let n be a positive integer and not a prime number. Then $\Gamma(Z_n)(\gamma) = K_{1,2(p-1)}$ if and only if $n = 3p$, where $p > 3$ is a prime number.

Proof: Assume that $n = 3p$, where $p \neq 2$ is a prime number.

Then $\Gamma(Z_n)(\gamma) = K_{2,(p-1)}$ and so $\gamma(\Gamma(Z_n)) = 2$.
Also, the number of $\gamma -$ sets is 2(p-1)+1.

Join the vertices that satisfy the two gamma graph $\Gamma(Z_n)(\gamma)$ conditions.
Then we get , $\Gamma(Z_n)(\gamma) = K_{1,2(p-1)}$.

The converse part is trivial.

Theorem 2.5: Let n be a positive integer and not a prime number. Then $\Gamma(Z_n)(\gamma) = K_{p-1}$ if and only if $n = p^k$ or $n = 2^{k_1}p^{k_2}$, where p is a prime number, $p > 3$ and $k \geq 2, k_1, k_2 > 1$.

Proof: Assume that $n = p^k$ or $n = 2^{k_1}p^{k_2}$, where p is a prime number, $p > 3$ and $k \geq 2, k_1, k_2 > 1$.

If $n = p^k$, $k > 2$, then $\{\mu p^{k-1}\}$ for $1 \leq \mu \leq p-1$ is a minimal dominating set in $\Gamma(Z_n)$.

Hence $\gamma(\Gamma(\mathbb{Z}_n)) = 1$.

By remark 1.3, $|V| = p - 1$.

Thus, $\Gamma(Z_n)(\gamma) = K_{p-1}$

If $n = 2^{k_1}p^{k_2}$, $k_1, k_2 > 1$, by remark 1.3, the number of $\gamma -$ sets are $p - 1$.

That is., $|V| = p - 1$.

Here each dominating sets satisfies the the gamma graph $\Gamma(Z_n)(\gamma)$ conditions.
So we can make each vertices adjacent to one another.
Thus, $\Gamma(Z_n)(\gamma) = K_{p-1}$
Conversely, Assume that $\Gamma(Z_n)(\gamma) = K_{p-1}$
Then the number of $\gamma -$ sets is $p - 1$.

By Remark 1.3, $r = 1$ and n is not a prime.

Hence $n = p^k$, $k \geq 2$, where p is a prime number, $p > 3$.

Now, Suppose $n = {p_1}^{k_1}{p_2}^{k_2}$, $k_1, k_2 > 1$ and $p_1 > 2$.

Which is obviously a contradiction.
Therefore $p_1 = 2$.

Hence $n = p^k$ or $n = 2^{k_1}p^{k_2}$, where p is a prime number,

$p > 3$ and $k \geq 2, k_1, k_2 > 1$.

3. Gamma Graph $\gamma.\Gamma(Z_n)$ of commutative ring R

Definition 3.1:[6] Let R be a commutative ring. The gamma graph of the zero-divisor graph $\Gamma(R)$ of R is the graph $\gamma.\Gamma(R)$ with vertex set as the collection of all gamma sets of the zero-divisor graph $\Gamma(R)$ of R and two distinct vertices A and B are adjacent if and only if $|A \cap B| = \gamma(\Gamma(R))$-1

Example 3.2: Let$R = Z_2 \times Z_2 \times Z_2$. Clearly,

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$V(\Gamma(R)) = \{u_1 = (1,0,0), u_2 = (0,1,0), u_3 = (0,0,1), v_1 = (0,1,1), v_2 = (1,0,1), v_3 = (1,1,0)\}$

Then the zero-divisor graph $\Gamma(R)$:

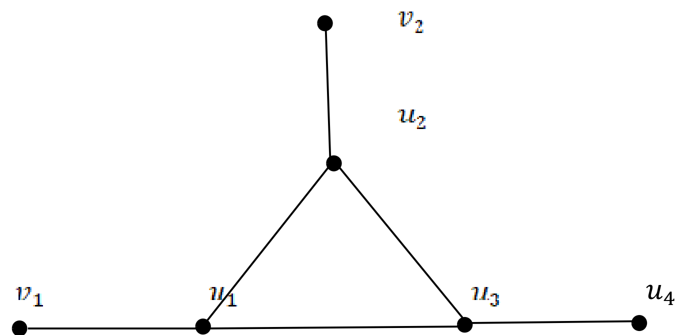

Fig.3 $\Gamma(R)$

Here, there are 8 $\gamma -$ sets in $\Gamma(R)$ and hence $V(\gamma.\Gamma(R)) = \{x_1 = \{u_1, u_2, u_3\}, x_2 = \{v_1, v_2, v_3\}, x_3 = \{u_1, v_2, v_3\}, x_4 = \{v_1, u_2, v_3\}, x_5 = \{v_1, v_2, u_3\}, x_6 = \{v_1, u_2, u_3\}, x_7 = \{u_1, v_2, u_3\}, x_8 = \{u_1, u_2, v_3\}$

Then the gamma graph $\gamma.\Gamma(R)$ is of the form:

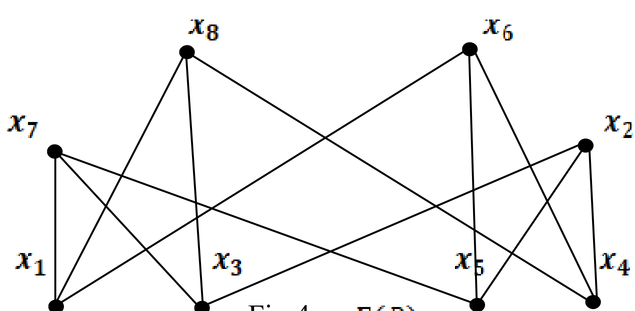

Fig 4. $\gamma.\Gamma(R)$

Theorem 3.3: Let (R, m) be a finite commutative local ring with $|R| = p^n$ for some prime $p,n > 1$. Then $\gamma.\Gamma(R) = K_{p^k-1}$, for some $k < n$.

Proof: Let $I = \bigcap_{x \in m^*} ann(x)$ be an ideal in R and $|I| = p^n$ for some $k < n$.

By proposition 1.7, $\gamma(\Gamma(R) = 1$ and for each $y \in I^*, ann(y) = m$ and $\{y\}$ is a $\gamma -$ set of $\Gamma(R)$

Thus, the number of $\gamma -$ sets $\Gamma(R)$ is $p^k - 1$ and intersection of any two $\gamma -$ sets is empty.

Therefore, $\gamma.\Gamma(R) = K_{p^k-1}$, for some $k < n$.

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4. Conclusions

In this paper, we have to find some results on Gamma Graph of a Zero-divisor graph $\Gamma(Z_n)(\gamma)$ and $\gamma.\Gamma(R)$. Further, we can extend these concept to total graph, order graph, comaximal graph etc…

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